Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

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3 **(a)** Find, in ascending powers of x, the first three terms in the expansion of $(2-x)^7$. [2]

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$(2-x)^7 = {}^7C_0(2)^7(-x)^0 + {}^7C_1(2)^6(-x)^1 + {}^7C_2(2)^5(-x)^2 + \ldots$

$= 1(128)(1) + 7(64)(-x) + 21(32)(x^2) + \ldots$

$= 128 - 448x + 672x^2 + \ldots$

$128 - 448x + 672x^2$ ans.

Hence, find the value of the constant a for which the coefficient of x^2 in the expansion of $(a-x)(2-x)^7$ is 616. [3]

$(a-x)(128-448x+672x^2) = \ldots + \underline{616}x^2 + \ldots$

$672ax^2 + 448x^2 = 616x^2$

$\therefore 672a + 448 = 616$

$672a = 168$

$a = \frac{1}{4}$

$a = \frac{1}{4}$ ans

(b) In the expansion of $\left(x^2-\frac{1}{2x^4}\right)^n$ in descending powers of x, the sixth term is independent of x. Find the value of n and the term independent of x. [4]

6th term $= {}^nC_5(x^2)^{n-5}\left(-\frac{1}{2x^4}\right)^5 = ___ x^0$

$= {}^nC_5(x^{2n-10})\left(-\frac{1}{2^5x^{20}}\right) = ___ x^0$

$= {}^nC_5\left(-\frac{1}{2^5}\right)x^{2n-10-20} = ___ x^0$

$\therefore 2n-30 = 0$

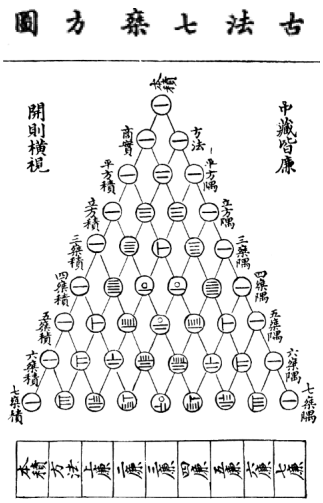
$n = 15$

$\therefore 6^{th}$ term $= {}^{15}C_5(x^2)^{10}\left(-\frac{1}{2x^4}\right)^5$

$= 3003\,x^{20}\left(-\frac{1}{32x^{20}}\right)$

$= -\frac{3003}{32}$

Yang Hui's Triangle (1238-1298)



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7 **(i)** Write down and simplify the first three terms in the expansion, in descending powers of x, of $\left(1-\frac{2}{x}\right)^8$. [2]

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$(1-\frac{2}{x})^8 = {}^8C_0(1)^8(-\frac{2}{x})^0 + {}^8C_1(1)^7(-\frac{2}{x})^1 + {}^8C_2(1)^6(-\frac{2}{x})^2 + \ldots$

$= 1 + 8(-\frac{2}{x}) + 28(-\frac{2}{x})^2 + \ldots$

$= 1 - \frac{16}{x} + \frac{112}{x^2}$

$1 - \frac{16}{x} + \frac{112}{x^2}$ Ans.

(ii) Given that there is no x term in the expansion of $\left(1-2x-kx^2\right)\left(1-\frac{2}{x}\right)^8$, find the constant term in the expansion. [4]

$(1-2x-kx^2)(1-\frac{16}{x}+\frac{112}{x^2}) = \ldots + 0x + \ldots$ for no x term

Using $1(-2x) + (-kx^2)(-\frac{16}{x})$, we have the coefficients

$-2 + 16k = 0$

$k = \frac{1}{8}$

For constant term, $(x^0) = 1(1) + (-2)(-16) + (-\frac{1}{8})(112)$

$= 1 + 32 - 14$

$= 19$ Ans.

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5 (i) Write down, and simplify, the first 4 terms in the expansion $(1+x)^7$ in ascending powers of x. [2]

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$(1+x)^7 = {}^7C_0(1)^7(x)^0 + {}^7C_1(1)^6(x)^1 + {}^7C_2(1)^5(x)^2 + {}^7C_3(1)^4(x)^3 + \ldots$

$= 1 + 7x + 21x^2 + 35x^3 + \ldots$

(ii) Write down the general term in the binomial expansion of $\left(x^2 - \frac{2}{x^3}\right)^9$. [1]

$(r+1)^{th}\ term = {}^9C_r(x^2)^{9-r}\left(-\frac{2}{x^3}\right)^r$

$= {}^9C_r(x^{18-2r})\frac{(-2)^r}{x^{3r}}$

$= {}^9C_r(-2)^r(x^{18-5r})$

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(iii) Write down the power of x in this general term. [1]

Power = $(18-5r)$

(iv) Hence, or otherwise, determine the coefficient of x^3 in the expansion of

$$(1+x)^7 + \left(x^2 - \frac{2}{x^3}\right)^9.$$ [2]

from part i) coefficient of x^3 for $(1+x)^7 = 35$

from part ii) ${}^9C_r(-2)^r(x^{18-5r})$ where $18-5r=3$

$\therefore r=3$

coefficient $x^3 = {}^9C_3(-2)^3 = 84(-(8)) = -672$

$\therefore$ coefficient $= 35 - 672 = -637$

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9. **(i)** Write down the general term in the binomial expansion of $\left(2x-\frac{1}{3x^3}\right)^6$. **[1]**

$(r+1)^{th}$ term $= {}^6C_r (2x)^{6-r}\left(-\frac{1}{3x^3}\right)^r$

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$= {}^6C_r (2^{6-r})\left(-\frac{1}{3}\right)^r (x^{6-r})\left(\frac{1}{x^{3r}}\right)$

$= {}^6C_r (2^{6-r})\left(-\frac{1}{3}\right)^r x^{6-4r}$

(ii) Explain, showing your working clearly, if there is a term independent of x in the expansion. **[2]**

for term independent of x (x^0), $6-4r = 0$

$6 = 4r$

$r = \frac{3}{2}$

∴ Since r is not an integer, there is no term independent of x.

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(iii) Using the general term, or otherwise, determine the term independent of x in the binomial expansion of $\left(3x^2+1+\frac{2}{x^2}\right)\left(2x-\frac{1}{3x^3}\right)^6$. **[4]**

For coefficient of x^{-2},

$$\left({}^6C_r(2^{6-r})\left(-\tfrac{1}{3}\right)^r x^{6-4r}\right) \quad \text{where} \quad 2+6-4r=0$$

$$r=2$$

$$=\left({}^6C_2(2^4)\left(-\tfrac{1}{3}\right)^2(x^{-2})\right)$$

$$=(15)(16)\left(\tfrac{1}{9}\right)x^{-2}$$

$$=26\tfrac{2}{3}x^{-2}$$

For coefficient of x^2,

$$\left({}^6C_r(2^{6-r})\left(-\tfrac{1}{3}\right)^r x^{6-4r}\right) \quad \text{where} \quad -2+6-4r=0$$

$$r=1$$

$$=\left({}^6C_1(2^5)\left(-\tfrac{1}{3}\right)^1\right)x^2$$

$$=(6)(32)\left(-\tfrac{1}{3}\right)x^2$$

$$=-64x^2$$

Hence, $\left(3x^2+1+\frac{2}{x^2}\right)\left(\ldots+26\tfrac{2}{3}x^{-2}-64x^2+\ldots\right)$

$\therefore$ the coefficient independent of x is $80-128$

$$=-48 \quad \text{Ans.}$$

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7 (i) Write down and simplify the first 3 terms in the expansion of $(1 - 3x)^5$ in ascending powers of x. [2]

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$(1-3x)^5 = {}^5C_0(1)^5(-3x)^0 + {}^5C_1(1)^4(-3x)^1 + {}^5C_2(1)^3(-3x)^2 + \ldots$

$= 1 + 5(-3)x + 10(-3)^2x^2 + \ldots$

$= 1 - 15x + 90x^2 + \ldots$

(ii) Given that the coefficient of x^2 in the expansion of $(1 - 3x)^5(1 + 5x)^n$ is 90, find the value of n, where n is a positive integer. [5]

using $(1+5x)^n = {}^nC_0(1)^n(5x)^0 + {}^nC_1(1)^{n-1}(5x)^1 + {}^nC_2(1)^{n-2}(5x)^2 + \ldots$

$= 1 + \frac{n}{1}(5)x^1 + \frac{n.(n-1)}{1.2}(5)^2x^2 + \ldots$

$(1-3x)^5(1+5x)^n$

$= (1 - 15x + 90x^2 + \ldots)\left(1 + \frac{n}{1}(5)x^1 + \frac{n.(n-1)}{1.2}(5)^2x^2 + \ldots\right)$

for coefficient of x^2, $\frac{n.(n-1)}{2}(25).1 + 5n.(-15) + 90(1) = 90$

$(n^2 - n)(25) - 150n + 180 = 90$

$25n^2 - 25n - 150n = 0$

$25n^2 - 175n = 0$

$n = 7$ or $n = 0$ (reject)

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2 (i) Write down, and simplify, the general term in the binomial expansion of $\left(x^2+\frac{3}{x}\right)^{15}$. [2]

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$(r+1)^{th}$ term $= {}^{15}C_r (x^2)^{15-r}\left(\frac{3}{x}\right)^r$

$= {}^{15}C_r (x^{30-2r})\left(\frac{3^r}{x^r}\right)$

$= {}^{15}C_r (3^r)(x^{30-3r})$

(ii) Hence determine the coefficient of x^3 in the binomial expansion of $\left(x^2+\frac{3}{x}\right)^{15}$. [3]

for x^3, we have $30-3r=3$

$r=9$

Hence, general term for coefficient $x^3 = {}^{15}C_9 (3^9)$

$= 98,513,415$ Ans

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2 The first three terms of the expansion of $(1 + ax)^n$ in ascending powers of x are $1 + 15x + 90x^2$.

(i) Find the values of a and n. [5]

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$(1+ax)^n = 1 + {}^nC_1(1)^{n-1}(ax)^1 + {}^nC_2(1)^{n-2}(ax)^2$

$= 1 + \frac{n}{1}(a)x + \frac{n(n-1)}{1\cdot 2}(a)^2 x^2$

comparing $(1+ax)^n$ with $1+15x+90x^2$, we have,

$an = 15$ — ①

$\frac{n(n-1)}{2}(a)^2 = 90$ — ②

sub ① into ②,

$\frac{n(n-1)}{2}\left(\frac{15}{n}\right)^2 = 90$

$(n^2-n)15^2 = 90(2)n^2$

$15(n^2-n) = 2n^2$

$15n^2 - n - 2n^2 = 0$

$13n^2 - n = 0$

$n = 3$ or $n = 0$ (reject)

$\therefore a = 5$

(ii) Find the constant term in the expansion of $\left(2 - \frac{1}{3x}\right)^2(1 + ax)^n$. [3]

$\left(2-\frac{1}{3x}\right)^2 = 4 - 2(2)\left(-\frac{1}{3x}\right) + \frac{1}{9x^2}$

$= 4 - \frac{4}{3x} + \frac{1}{9x^2}$

$(1+ax)^n = 1 + \frac{n}{1}(a)x + \frac{n(n-1)}{1\cdot 2}(a)^2 x^2 \rightarrow 1 + \frac{3}{1}(5)x + \frac{3(2)}{1\cdot 2}(5)^2 x^2$

$(1+5x)^3 = 1 + 15x + 75x^2$

$\therefore \left(4 - \frac{4}{3x} + \frac{1}{9x^2}\right)(1 + 15x + 75x^2)$

Constant term $= 4(1) + \left(-\frac{4}{3}\right)(15) + \left(\frac{1}{9}\right)(75)$

$=$

Ans.

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